Mid-Semestral Exam
Algebra-I
B. Math - First year
2014-2015

Time: 3 hrs
Max score: 100

Answer all questions.

(1) State true or false. Justify your answers.
(a) If G is a finite group with normal subgroups N_1 and N_2 such that $N_1 \cong N_2$, then $G/N_1 \cong G/N_2$.
(b) $Z(S_n)$, the centre of the symmetric group of degree n, is trivial for $n \geq 3$.
(c) If $G/Z(G)$ is cyclic, then G is abelian.
(d) Any abelian group of order 21 is cyclic.
(e) Any element in S_n $n \geq 2$ can be written as a product of transpositions of the form $(1\ k)$ where $k = 2, \ldots, n$. (5+5+5+5+5)

(2) Investigate the group generated by the following 2×2 matrices with complex entries. Find the order of the group, normal subgroups and quotients.
$A= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ (15)

(3) (a) Let G be a cyclic group of finite order n and let m be a divisor of n. Show that G has a unique subgroup of order m.
(b) Show that the multiplicative group $(\mathbb{Z}/2^n\mathbb{Z})^{\times}$, of all multiplicative inverses of $\mathbb{Z}/2^n\mathbb{Z}$, is not cyclic for any $n \geq 3$. (7+8)

(4) Let D_{2n} be the dihedral group of order $2n$. Show that
(a) $Z(D_{2n}) = \{1\}$ if n is odd
(b) $Z(D_{2n}) = \{1, r^k\}$ if $n = 2k$. (15)

(5) Let $GL_2(\mathbb{R})$ denote the multiplicative group of all 2×2 invertible matrices over $\mathbb{R}$. Consider the mutiplicative action of $GL_2(\mathbb{R})$ on $\mathbb{R}^2$.
(i) Describe the orbit of this action containing the vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$.
(ii) Describe the other orbits of this action. (10)

(6) (i) Let G be a group and let H be a subgroup of G. Consider the action of G on the set X of all left cosets of H in G by left multiplication. Determine the kernel of the action. Show that the kernel is the largest normal subgroup of G contained in H.

(ii) Prove that if H has finite index n, then there is a normal subgroup K of G with $K \subseteq H$ and $|G : K| \leq n!$. (10+10)